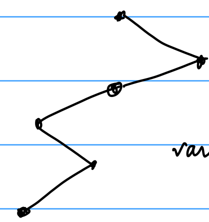
Ehrenfest Urn / osmosis

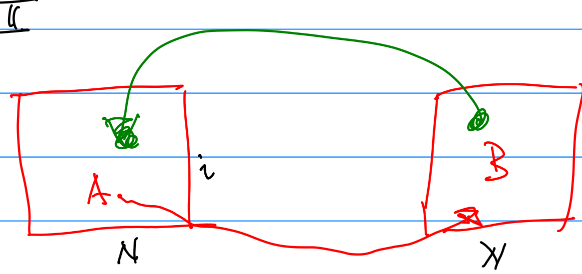
Ornstein - Uhlenbeck



Brownian motion

O-U ~ random walk with restoring force

Karlin + Taylor II



probability of $i+1$ $= \frac{2N-i}{2N} = 1 - \frac{i}{2N}$ occupation of urn A

probability of $i-1$ $= \frac{i}{2N} = \frac{i}{2N}$

Write as Markov chain with matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & \cdots & & & \\ \frac{1}{2N} & 0 & 1-\frac{1}{2N} & 0 & \cdots & & \\ 0 & \frac{1}{2} & 0 & 1-\frac{1}{2N} & 0 & \cdots & \\ & & & & & & \end{pmatrix}$$ ergodic

stationary state $p_{eq} = \left(\frac{1}{N}, \ldots\ldots, \frac{1}{N}\right)$

τ time between transitions

$X(t) = X_N(t)$ = # balls in A at time t

$\Delta X_N = X_N(t+\tau) - X_N(t)$

$Pr\{\Delta X = +1 \mid X(t) = x\} = \frac{1}{2} + \frac{N-x}{2N}$ $\left(= \frac{2N-x}{2N}\right)$

$Pr\{\Delta X = -1 \mid X(t) = x\} = \frac{1}{2} - \frac{N-x}{2N}$ $\left(= \frac{x}{2N}\right)$

$N \to \infty$, $\tau \to 0$ with $N\tau = 1$

measure fluctuations in units of $\frac{1}{\sqrt{N}}$

$$Y(t) = Y_N(t) = \frac{1}{\sqrt{N}} \left(X([Nt]) - N \right)$$

unit of time $\rightsquigarrow$ N transitions of original process

unit change $\rightsquigarrow$ fluctuations of order $\frac{1}{\sqrt{N}}$

$$\Delta Y_N(t, \tau) = Y_N(t+\tau) - Y_N(t) \qquad \tau = \frac{1}{N}$$

$$\Pr\left\{ \Delta Y = \pm \frac{1}{\sqrt{N}} \,\middle|\, Y_N(t) = y \right\} = \Pr\left\{ \Delta X = \pm 1 \,\middle|\, X_N([Nt]) = x = N + y\sqrt{N} \right\}$$

$$= \frac{1}{2} \pm \frac{N - (N + y\sqrt{N})}{2N}$$

$$= \frac{1}{2} \mp \frac{y}{2\sqrt{N}}$$

for $\{ Y_N(t) \}$ compute OU

$$E\left(\Delta Y \,|\, Y_N(0) = y \right) = \frac{1}{\sqrt{N}} \left(\frac{1}{2} - \frac{y}{2\sqrt{N}} \right) - \frac{1}{\sqrt{N}} \left(\frac{1}{2} + \frac{y}{2\sqrt{N}} \right)$$

$$= -\frac{y}{N} = -y\tau$$

$\frac{1}{\tau} E\left(\Delta Y \,|\, Y_N(0) = y \right) \to -y$ as $\tau \to 0$ (ie as $N \to \infty$)

$$\frac{1}{\tau} E\left(|\Delta Y|^2 \,|\, Y_N(0) = y \right) = \frac{1}{N\tau} = 1$$

$$\mu = \frac{1}{h} E\left(\Delta_h X(t) \,|\, X(t) = x \right)$$

$$D = \frac{1}{h} E\left(|\Delta_h X(t)|^2 \,|\, X(t) = x \right)$$

$$dX = \mu X \, dt + \sqrt{D} \, dB$$

$\Rightarrow$ forward and backward Kolmogorov Equations

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} \left(D \frac{\partial \rho}{\partial x} + \mu \rho \right) \qquad \text{forward}$$

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\sigma^2}{2}\frac{\partial \rho}{\partial x} - x\rho\right)$$ O-U equation

found for our statistic